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STRINGS AND SUPERSYMMETRY AS TOOLS FOR PERTURBATIVE QCD*

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ABSTRACT

We review techniques simplifying the analytic calculation of one-loop QCD amplitudes with many external legs, for use in next-to-leading-order corrections to multi-jet processes. We explain how a supersymmetry-inspired organization works well in conjunction with other tools, namely the color and helicity decompositions of amplitudes, and the constraints imposed by perturbative unitarity and collinear singularities. String theory seems most useful as a heuristic guide. Using these techniques, the complete set of one-loop five-parton QCD amplitudes, as well as certain sequences of special helicity amplitudes with an arbitrary number of external gluons, have been obtained.

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1. Motivation

Although most people at this conference may be convinced that supersymmetry and superstring theory are realized in Nature, let us suppose for the sake of argument that they are not. Can these beautiful theories then have any practical implications? The answer is yes; they can still help to organize complicated perturbative gauge theory calculations, particularly in QCD. We will argue that supersymmetry works best in conjunction with a number of other tools: the color and helicity decompositions of amplitudes, and the constraints imposed by perturbative unitarity and collinear singularities. At present, at least for one-loop calculations, string theory seems more useful as a heuristic guide to how to organize calculations, rather than as a detailed calculation tool.

Next-to-leading-order (NLO) perturbative QCD corrections are important for precision comparison of theoretical predictions with collider experiments, for many multi-jet and jet-associated processes. Currently, NLO results are only available for processes involving four "partons" (one or more of the partons may be replaced by a γ, Z or W), for example $p\bar{p} \to 2$ jets[1] and $e^+e^- \to 3$ jets.[2] There are two parts to an NLO correction to an n-parton process: a real (or tree) part, obtained by integrating the tree-level cross-section for a $(n+1)$-parton process over an "unobserved" portion of phase space; and a virtual (or one-loop) part, obtained by interfering the one-loop n-parton amplitude with the corresponding tree amplitude. Because the calculation of tree amplitudes is now fairly efficient,[3] it is the calculation of one-loop multi-parton amplitudes that forms the "analytical bottleneck" to producing NLO results for more complicated processes. (It is not the only obstacle, however; much numerical work is required to combine the real and virtual corrections.) The difficulty in going to more than four external partons is indicated by the time lag between the calculation of one-loop four-parton amplitudes in 1980[2] and 1986[4] and that of five-parton amplitudes in the past two years.[5,6,7]

In principle it is straightforward to compute one-loop amplitudes by drawing all Feynman diagrams and evaluating them using standard reduction techniques for the loop integrals. In practice this method becomes extremely inefficient and cumbersome as the number of external legs grows, because there are:

1. **too many diagrams** — many diagrams are related by gauge invariance, and
2. **too many terms in each diagram** — nonabelian gauge boson self-interactions are complicated.

Consequently, intermediate expressions tend to be vastly more complicated than the final results, when the latter are represented in an appropriate way.

A useful organizational framework, that helps tame the size of intermediate expressions, is Total Quantum-number Management (TQM), which suggests to:

- Keep track of all quantum numbers of external particles — namely, *helicity* and *color* information.
- Use the helicity/color information to decompose the amplitude into simpler, gauge-invariant pieces, called *primitive amplitudes*.

• Use supersymmetry to organize the sum over *internal* particle spins in the loop.
• Square amplitudes to get probabilities, and sum over helicities and colors to obtain unpolarized cross-sections, only at the very *end* of the calculation.
Carrying out the last step explicitly would generate a large analytic expression; however, at this stage one would typically make the transition to numerical evaluation, in order to combine the virtual and real corrections. The use of TQM is hardly new, particularly in tree-level applications[3] — but it is especially useful at loop level.

2. What about string theory?

What role can superstring theory play in this approach? String theory has the advantage that all field theory diagrams at a given order of perturbation theory are lumped into a single string diagram. Circulating in the loop(s) of the diagram are not just the quarks and gluons of QCD, but the entire massive tower of string excitations, with masses of order the Planck mass. In the experimentally relevant low-energy limit, the massive states decouple, and the string diagram degenerates into a number of field-theory-like diagrams, determined by simple (Bern-Kosower[8]) rules that are nicely compatible with the color and helicity decompositions.

However, there are also a few disadvantages to a direct string-based approach. Simple rules can still generate a large mess in intermediate steps. (This was found to be the case in the calculation of one-loop five-gluon amplitudes.) The Bern-Kosower rules were derived for one-loop amplitudes with external gluons only, and new rules would have to be rederived for external quarks and/or multi-loop amplitudes. Some progress has been made in these directions,[9] but not yet to the point of pushing the field theory state-of-the-art. On the other hand, string-based rules can be mimicked in field theory[10] by a combination of background-field[11] and Gervais-Neveu[12] gauges. Such gauge choices can be used with external fermions too. Finally, other tools — in particular supersymmetry, unitarity and collinear limits — can be even more efficient routes to one-loop scattering amplitudes. String theory remains useful as a heuristic guide; we will give a couple of examples below.

3. Color and helicity decomposition

As an example of the color and helicity decomposition of a one-loop QCD amplitude, consider the amplitude for n external gluons, all taken to be outgoing. We generalize the $SU(3)$ color group to $SU(N_c)$, and label the gluons $i = 1, 2, \ldots, n$ by their adjoint color indices $a_i = 1, 2, \ldots, N_c^2 - 1$, and helicities $\lambda_i = \pm$. Without giving the details of the helicity decomposition formalism, it is convenient to use gluon circular polarization vectors expressed in terms of massless Weyl spinors.[13] The color decomposition[14] is performed in terms of traces of $SU(N_c)$ generators T^a in the fundamental representation, with $\mathrm{Tr}(T^a T^b) = \delta^{ab}$,

$$\mathcal{A}_n^{1-\mathrm{loop}}\left(\{k_i, \lambda_i, a_i\}\right) = g^n \Bigg[\sum_{\sigma \in S_n/Z_n} N_c \, \mathrm{Tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}\right) \; A_{n;1}(\sigma(1^{\lambda_1}), \ldots, \sigma(n^{\lambda_n}))$$

$$+ \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \mathrm{Tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(c-1)}}\right) \; \mathrm{Tr}\left(T^{a_{\sigma(c)}} \cdots T^{a_{\sigma(n)}}\right) \quad A_{n;c}(\sigma(1^{\lambda_1}), \ldots, \sigma(n^{\lambda_n})) \Bigg], \quad (1)$$

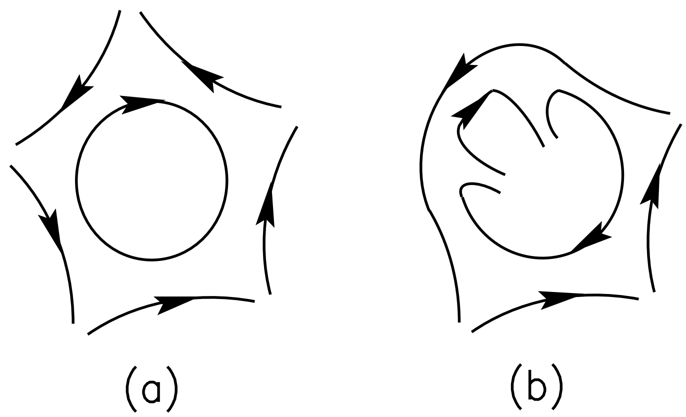

Figure 1: The two types of oriented one-loop open string world-sheet diagrams: Either (a) all vertices attach to the same edge of the annulus, or (b) some vertices attach to each edge.

where $A_{n;c}$ are the partial amplitudes, g is the gauge coupling, and $\lfloor x \rfloor$ is the integer part of x. S_n is the set of all permutations of n objects, while Z_n and $S_{n;c}$ are the subsets of S_n that leave the corresponding single and double trace structures invariant.

The color decomposition (1) can be derived fairly simply from Feynman diagrams in the double-line formalism;[15] however, it is even more transparent to derive it from open string theory. The end of an open string carries the color information — an index i in the fundamental N_c representation. A vertex operator for an external gluon with adjoint index a carries the Chan-Paton factor[16] $(T^a)_i^{\ \bar{\jmath}}$ and rotates the fundamental index. At tree-level, the string world-sheet is a topologically a disk, and the Chan-Paton factors hook together into a single trace of the form $\mathrm{Tr}\,(T^{a_1} \cdots T^{a_n})$ (or some permutation thereof). At one-loop, the (oriented) world-sheet is an annulus, and there are two possibilities, shown in fig. 1: either (a) all vertices attach to the same edge of the annulus, or (b) some vertices attach to each of the two edges. In case (a) one obtains the single-trace structure of the first term in eq. (1), which multiplies $A_{n;1}$; the factor of $N_c = \mathrm{Tr}(1)$ comes from the edge with no vertices attached to it. In the second case one obtains the remaining double-trace terms in (1), which multiply $A_{n;c>1}$.

Closer inspection of the two different types of string world-sheet diagrams in the low-energy limit *suggests* a possible relation between the corresponding $A_{n;c>1}$ and $A_{n;1}$ amplitudes. Apart from the fact that some vertex operators appear on the "wrong side" of the annulus — which one might hope is irrelevant in the low-energy limit — the $A_{n;c}$ world-sheet diagrams appear to be just the sum over a particular set $COP\{\alpha\}\{\beta\}$ of permutations of the $A_{n;1}$ diagrams, those that preserve the cyclic ordering of the sets $\{\alpha\}$ and $\{\beta\}$ of vertices on the respective inner and outer boundaries of the annulus. Thus the following formula is suggested,

$$A_{n;c}(1,2,\ldots,c-1;c,c+1,\ldots,n) \ = \ (-1)^{c-1} \sum_{\sigma \in COP\{\alpha\}\{\beta\}} A_{n;1}(\sigma), \tag{2}$$

where $\{\alpha\} \equiv \{c-1, c-2, \ldots, 2, 1\}$, $\{\beta\} \equiv \{c, c+1, \ldots, n-1, n\}$. This formula is of practical use because now only the $A_{n;1}$ have to be calculated. It can be proven using Feynman diagrams,[17] but its motivation from the structure of open string theory is a good example of how strings can serve as a heuristic guide to gauge theory organization.

Equation (2) shows that the $A_{n;1}$ are the more basic objects, so we call them *primitive amplitudes*. They are "color-ordered" amplitudes, in that they only receive contributions from diagrams with a particular cyclic ordering of the gluons around the loop. This greatly simplifies their analytic structure, because cuts and poles can only appear in channels formed by the sum of *cyclically adjacent* momenta, $(k_i + k_{i+1} + \cdots + k_{i+r-1})^2$.

Even the $A_{n;1}$ are not all independent, due to parity and cyclic invariance. For example, for $n = 5$ only four are independent, $A_{5;1}(1^+, 2^+, 3^+, 4^+, 5^+)$, $A_{5;1}(1^-, 2^+, 3^+, 4^+, 5^+)$, $A_{5;1}(1^-, 2^-, 3^+, 4^+, 5^+)$, and $A_{5;1}(1^-, 2^+, 3^-, 4^+, 5^+)$. The first two are not required at NLO because the corresponding tree helicity amplitudes vanish, and are very simple for the same reason. Analytic expressions for the latter two are more complex[5] but still "fit on a page" (see below). In contrast, the color- and helicity-summed virtual correction to the cross-section, built from permutation sums of the latter two primitive amplitudes, would fill hundreds of pages.

4. What about supersymmetry?

Supersymmetry plays a role even in a non-supersymmetric theory such as QCD. This is because tree-level QCD is "effectively" supersymmetric.[18] Consider the n-gluon tree amplitude. It has no loops in it, therefore it has no fermion loops in it. Therefore the fermions in the theory might as well be gluinos, i.e. at tree-level the theory might as well be super Yang-Mills theory. The "non-supersymmetry" of QCD only leaks in at the loop level.

Supersymmetric results are often simpler than non-supersymmetric ones. For example, the anomalous magnetic moments of the electron in QED and in super-QED are[19,20]

$$\begin{aligned} \left.\frac{g_e-2}{2}\right|_{\rm QED} &= \frac{\alpha}{2\pi} + \left[\frac{197}{144} + \frac{\pi^2}{12} + \frac{3\zeta_3}{4} - \frac{\pi^2 \ln 2}{2}\right]\left(\frac{\alpha}{\pi}\right)^2 + 1.1761(4)\left(\frac{\alpha}{\pi}\right)^3 - 1.43(14)\left(\frac{\alpha}{\pi}\right)^4 + \cdots \\ \left.\frac{g_e-2}{2}\right|_{\rm SQED} &= 0; \end{aligned} \tag{3}$$

the latter is an example of a supersymmetry Ward identity (SWI).[20] Here the supersymmetric result is "too simple": it does not form a significant part of the non-supersymmetric result. For one-loop multi-parton QCD calculations, the situation is somewhat more favorable.

Supersymmetry Ward identities[21] can be derived for general S-matrix elements $\langle \Phi_1 \cdots \Phi_n \rangle$ using the fact that the supercharge Q annihilates the vacuum; when the fields Φ_i create helicity eigenstates, many of the $[Q, \Phi_i]$ terms can be arranged to vanish. Taking all particles to be outgoing, the simplest identities are for amplitudes with at most two negative helicities, and the rest positive[21,18,3]:

$$A_n^{\rm SUSY}(1^\pm, 2^+, 3^+, \ldots, n^+) = 0, \tag{4}$$

$$A_n^{\rm SUSY}(1^-, 2_P^-, 3_P^+, 4^+, \ldots, n^+) = \left(\frac{\langle 1\,2\rangle}{\langle 1\,3\rangle}\right)^{2|h_P|} A_n^{\rm SUSY}(1^-, 2_\phi^-, 3_\phi^+, 4^+, \ldots, n^+). \tag{5}$$

Here ϕ stands for a scalar particle (for which the "helicity" $\pm$ means particle vs. antiparticle), while P stands for a scalar, fermion or gluon, with respective helicity $h_P = 0, \frac{1}{2}, 1$. We have

introduced spinor product notation,[13,3] $\langle j\,l\rangle = \langle j^-|l^+\rangle = \bar{u}_-(k_j)u_+(k_l)$ and $[j\,l] = \langle j^+|l^-\rangle = \bar{u}_+(k_j)u_-(k_l)$, where $u_\pm(k)$ is a massless Weyl spinor with momentum k and chirality $\pm$.

The SWI hold order-by-order in perturbation theory. They apply directly to all *tree-level* QCD amplitudes because of the "effective" supersymmetry described above. They guide the simple structure of "maximally helicity violating" (MHV) QCD tree amplitudes, which for n external gluons are[22,23]

$$A_n^{\rm tree}(1^\pm,2^+,3^+,\ldots,n^+) \;=\; 0, \tag{6}$$

$$A_n^{\rm tree}(1^+,\ldots,j^-,\ldots,k^-,\ldots,n^+) \;=\; i\,\frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\cdots\langle n\,1\rangle}\,. \tag{7}$$

Even in the second, nonvanishing case, the amplitude remains simple because the SWI forbid the appearance of multi-particle poles (poles in $(k_i+\cdots+k_{i+r-1})^2$ with $r>2$). The intermediate gluon in the factorization of eq. (7) on a multi-particle pole has negative helicity as seen by one of the two lower-point amplitudes, but positive helicity as seen by the other. Thus the two lower-point amplitudes share a total of three negative helicities, and so one of them must vanish by eq. (4).

At loop level, QCD "knows" that it is not supersymmetric. However, one can use supersymmetry to trade gluons in the loop diagrams for scalars. Scalars lead to algebraically simpler diagrams, because they cannot propagate spin information around the loop. For an amplitude with all external gluons, we rewrite the internal gluon loop g (and fermion loop f) as a supersymmetric contribution plus a complex scalar loop s,

$$\begin{aligned} g &= (g+4f+3s) - 4(f+s) + s = A^{N=4} - 4\,A^{N=1} + A^{\rm scalar},\\ f &= (f+s) - s = A^{N=1} - A^{\rm scalar}, \end{aligned} \tag{8}$$

where $A^{N=4}$ represents the contribution of the $N=4$ super Yang-Mills multiplet, and $A^{N=1}$ an $N=1$ chiral matter supermultiplet. In the context of TQM, this use of supersymmetry could be termed "internal spin management".

As an example, let's look at the five-gluon primitive amplitude $A_{5;1}(1^-,2^-,3^+,4^+,5^+)$, whose components according to (8) are[5]

$$\begin{aligned}
A^{N=4} &= c_\Gamma\, A^{\rm tree}\sum_{j=1}^{5}\left[-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{j,j+1}}\right)^{\epsilon} + \ln\left(\frac{-s_{j,j+1}}{-s_{j+1,j+2}}\right)\ln\left(\frac{-s_{j+2,j-2}}{-s_{j-2,j-1}}\right) + \frac{\pi^2}{6}\right]\\
A^{N=1} &= c_\Gamma\, A^{\rm tree}\left[\frac{5}{2\epsilon} + \frac{1}{2}\left[\ln\left(\frac{\mu^2}{-s_{23}}\right) + \ln\left(\frac{\mu^2}{-s_{51}}\right)\right] + 2\right]\\
&\quad + \frac{ic_\Gamma}{2}\frac{\langle 1\,2\rangle^2\left(\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle + \langle 2\,4\rangle\,[4\,5]\,\langle 5\,1\rangle\right)}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\frac{\ln\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}-s_{23}}\\
A^{\rm scalar} &= \frac{1}{3}A^{\rm N=1} + \frac{2}{9}c_\Gamma\, A^{\rm tree}\\
&\quad + \frac{ic_\Gamma}{3}\Bigg[-\frac{[3\,4]\,\langle 4\,1\rangle\,\langle 2\,4\rangle\,[4\,5]\left(\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle + \langle 2\,4\rangle\,[4\,5]\,\langle 5\,1\rangle\right)}{\langle 3\,4\rangle\,\langle 4\,5\rangle}\frac{\ln\left(\frac{-s_{23}}{-s_{51}}\right) - \frac{1}{2}\left(\frac{s_{23}}{s_{51}} - \frac{s_{51}}{s_{23}}\right)}{(s_{51}-s_{23})^3}\\
&\quad - \frac{\langle 3\,5\rangle\,[3\,5]^3}{[1\,2]\,[2\,3]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[5\,1]} + \frac{\langle 1\,2\rangle\,[3\,5]^2}{[2\,3]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[5\,1]} + \frac{1}{2}\frac{\langle 1\,2\rangle\,[3\,4]\,\langle 4\,1\rangle\,\langle 2\,4\rangle\,[4\,5]}{s_{23}\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,s_{51}}\Bigg]\,,
\end{aligned} \tag{9}$$

where $A^{\rm tree} = A_5^{\rm tree}(1^-,2^-,3^+,4^+,5^+)$ is given in eq. (7), and

$$c_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)}\ . \tag{10}$$

We see that the three components have quite different analytic structure, indicating that the rearrangement (8) is a natural one. The $N=4$ supersymmetric component is the simplest, followed by the $N=1$ chiral component. The non-supersymmetric scalar component is the most complicated, and the hardest to calculate. Yet it is still simpler than the direct gluon calculation, because it does not mix all three components together.

5. One-loop amplitudes via unitarity

The absorptive parts (cuts) of loop amplitudes can be determined from phase-space integrals of products of lower-order amplitudes, exploiting the perturbative unitarity of the S-matrix. For one-loop multi-parton amplitudes, there are several reasons why this calculation of the cuts is much easier than a direct loop calculation:

• One can simplify the tree amplitudes *before* feeding them into the cut calculation.
• The tree amplitudes are usually quite simple, because they possess "effective" supersymmetry, even if the full loop amplitudes do not.
• One can further use on-shell conditions for the intermediate legs in evaluating the cuts.

The catch is that it is not always possible to reconstruct the full loop amplitude from its cuts. In general there can be an additive "polynomial ambiguity" — in addition to the usual logarithms and dilogarithms of loop amplitudes, there may be polynomials (actually rational functions) in the kinematic variables, which cannot be detected by the cuts. This ambiguity is absent in one-loop massless supersymmetric amplitudes,[17,24] because of their better ultraviolet behavior. Notice that in the five-gluon example (9) all the polynomial terms are intimately linked to the logarithms in both $A^{N=4}$ and $A^{N=1}$, while they are not linked in $A^{\rm scalar}$.

To see the supersymmetric cancellations for n-gluon amplitudes, it suffices to use the second-order formalism for the fermion loop, and background-field gauge[11] for the gluon loop, in the effective action $\Gamma(A)$. The scalar, fermion and gluon contributions are

$$\begin{aligned} \Gamma^{\rm scalar}(A) &\sim +\ln\det\Big(D^2\Big)\,, \\ \Gamma^{\rm fermion}(A) &\sim -\ln\det\Big(D^2 - \tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}\Big)\,, \\ \Gamma^{\rm gluon}(A) &\sim +\ln\det\Big(D^2 - \Sigma^{\mu\nu}F_{\mu\nu}\Big)\,, \end{aligned} \tag{11}$$

where D is the covariant derivative, F is the external field strength, and $\frac{1}{2}\sigma_{\mu\nu}$ ($\Sigma_{\mu\nu}$) is the spin-$\frac{1}{2}$ (spin-1) Lorentz generator. The leading behavior for large loop-momentum ℓ comes from the D^2 term in each case ($F_{\mu\nu}$ contains derivatives only with respect to the external momenta). This term cancels between the scalar and fermion, and between the fermion and gluon in eq. (11), hence it cancels in any supersymmetric linear combination. The cancellation for an m-point graph is from ℓ^m down to ℓ^{m-2} (since $\mathrm{Tr}\,\sigma_{\mu\nu} = \mathrm{Tr}\,\Sigma_{\mu\nu} = 0$). It can be shown that an amplitude having this property in all graphs can be uniquely reconstructed from its cuts.[24]

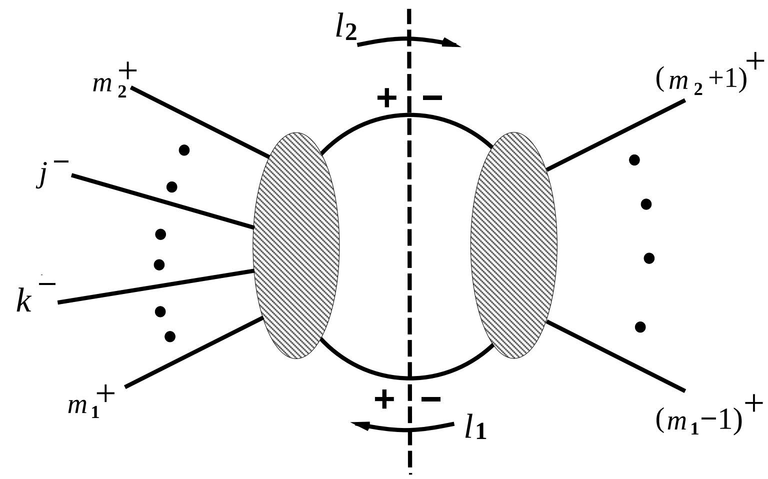

Figure 2: The possible intermediate helicities when both negative helicity gluons lie on the same side of the cut.

The Neveu-Schwarz-Ramond representation of superstring theory, although not manifestly supersymmetric, corresponds to both a second-order fermion and background-field gauge formalism in field theory. This is a second example of string theory as a heuristic guide.

Non-supersymmetric amplitudes generally cannot be directly reconstructed from their unitarity cuts. We did not discuss the collinear behavior of loop amplitudes here,[25,17,26] but they are a useful and powerful practical tool for fixing the polynomial ambiguities, recursively in n, by requiring consistent collinear factorization in all channels. The only drawback is the current lack of a theorem that would guarantee the uniqueness of a polynomial expression obeying all collinear consistency checks. But no counterexamples are known either, for $n > 5$. It is also possible to extract $\mathcal{O}(\epsilon^0)$ polynomial terms from cuts evaluated to $\mathcal{O}(\epsilon)$ in dimensional regularization, but this is significantly harder than evaluation of the cuts to $\mathcal{O}(\epsilon^0)$.

As an example of how simple one-loop multi-parton cuts can be, we outline here the evaluation of the cuts for an infinite sequence of n-gluon amplitudes, the MHV amplitudes in $N = 4$ super-Yang-Mills theory.[17] We consider the case where the two negative helicity gluons lie on the same side of the cut, as shown in fig. 2. (The case where they lie on the opposite side of the cut can be quickly reduced to this case[17] using the SWI (5).) Contributions to this cut from intermediate fermions or scalars vanish using the "effective" supersymmetry of the tree amplitudes, eq. (4), and the conservation of fermion helicity and scalar particle number. The only contribution is from intermediate gluons with the helicity assignment shown in fig. 2. The tree amplitudes on either side of the cut are pure-glue MHV tree amplitudes, given in eq. (7).

Let j and k denote the negative helicity external gluons. The cut for this MHV loop amplitude, $A_{jk}^{1-\text{loop MHV}}(1,2,\ldots,n)$, in the channel $(k_{m_1}+k_{m_1+1}+\cdots+k_{m_2-1}+k_{m_2})^2$, where $m_1 \le k < j \le m_2$, is then given by

$$\begin{aligned}
&\int d\text{LIPS}(-\ell_1,\ell_2) A_{jk}^{\text{tree MHV}}(-\ell_1,m_1,\ldots,m_2,\ell_2)\; A_{(-\ell_2)\ell_1}^{\text{tree MHV}}(-\ell_2,m_2+1,\ldots,m_1-1,\ell_1) \\
=\;& i\,A_{jk}^{\text{tree MHV}}(1,2,\ldots,n)\int d\text{LIPS}(-\ell_1,\ell_2)\frac{\langle (m_1-1)\,m_1\rangle\,\langle \ell_1\,\ell_2\rangle}{\langle (m_1-1)\,\ell_1\rangle\,\langle \ell_1\,m_1\rangle}\frac{\langle m_2\,(m_2+1)\rangle\,\langle \ell_2\,\ell_1\rangle}{\langle m_2\,\ell_2\rangle\,\langle \ell_2\,(m_2+1)\rangle}\;, \quad (12)
\end{aligned}$$

where the spinor products are labelled by either loop momenta (ℓ_1, ℓ_2) or external particle labels. The $(4-2\epsilon)$-dimensional Lorentz-invariant phase space measure is denoted by $d\mathrm{LIPS}(-\ell_1, \ell_2)$. The cut (12) remains simple for arbitrarily many external gluons, thanks to the simple form of the MHV tree amplitudes (7) — ℓ_1 and ℓ_2 appear in only a few of the factors.

The integral (12) can be viewed as a cut hexagon loop integral. To see this, use the on-shell condition $\ell_1^2 = \ell_2^2 = 0$ to rewrite the four spinor product denominators in (12) as scalar propagators, multiplied by a numerator factor. For example, $1/\langle \ell_1\, m_1\rangle = [m_1\, \ell_1]\,/(\langle \ell_1\, m_1\rangle\, [m_1\, \ell_1]) = [m_1\, \ell_1]\,/(2\ell_1\cdot k_{m_1}) = -\,[m_1\, \ell_1]\,/(\ell_1 - k_{m_1})^2$. In addition to these four propagators, there are two cut propagators implicit in the phase-space integral $\int d\mathrm{LIPS}(-\ell_1, \ell_2)$. The Schouten identity, $\langle a\, b\rangle\, \langle c\, d\rangle = \langle a\, d\rangle\, \langle c\, b\rangle + \langle a\, c\rangle\, \langle b\, d\rangle$, lets us rewrite the integrand of (12) as

$$\begin{aligned}
\mathcal{I} &= \Big(\frac{\langle (m_1-1)\,\ell_2\rangle}{\langle (m_1-1)\,\ell_1\rangle} - \frac{\langle m_1\,\ell_2\rangle}{\langle m_1\,\ell_1\rangle}\Big)\Big(\frac{\langle m_2\,\ell_1\rangle}{\langle m_2\,\ell_2\rangle} - \frac{\langle (m_2+1)\,\ell_1\rangle}{\langle (m_2+1)\,\ell_2\rangle}\Big) \\
&= -\frac{\langle m_1\,\ell_2\rangle\,\langle m_2\,\ell_1\rangle}{\langle m_1\,\ell_1\rangle\,\langle m_2\,\ell_2\rangle} \pm \Big[\; m_1 \leftrightarrow (m_1-1)\;,\; m_2 \leftrightarrow (m_2+1)\;\Big]\,,
\end{aligned} \tag{13}$$

antisymmetrizing in each exchange. In terms of propagators,

$$\begin{aligned}
\mathcal{I} &= \frac{[\ell_1\, m_1]\,\langle m_1\,\ell_2\rangle\,[\ell_2\, m_2]\,\langle m_2\,\ell_1\rangle}{(\ell_1 - k_{m_1})^2(\ell_2 + k_{m_2})^2} \pm \Big[\; m_1 \leftrightarrow (m_1-1)\;,\; m_2 \leftrightarrow (m_2+1)\;\Big] \\
&= \frac{\mathrm{tr}_+(\not{\ell}_1\; \not{k}_{m_1}\; \not{\ell}_2\; \not{k}_{m_2})}{(\ell_1 - k_{m_1})^2(\ell_2 + k_{m_2})^2} \pm \Big[\; m_1 \leftrightarrow (m_1-1)\;,\; m_2 \leftrightarrow (m_2+1)\;\Big]\,,
\end{aligned} \tag{14}$$

where the tr_+ indicates the insertion of a $(1+\gamma_5)/2$ projector into the trace. Thus we have reduced the cut hexagon integral (12) to a sum of four cut box integrals.

A straightforward Passarino-Veltman reduction[27] expresses the box integrals from (14) in terms of scalar boxes, triangles and bubbles. The coefficients of the triangles and bubbles vanish. The only scalar boxes with non-vanishing coefficients are those with two diagonally opposite massless legs. The full amplitude, which matches the cuts in all channels, is

$$A_{n;1}^{N=4}(1^+,\ldots,j^-,\ldots,k^-,\ldots,n^+) \;=\; i\, c_\Gamma\, \mu^{2\epsilon}\, \frac{\langle j\, k\rangle^4}{\langle 1\, 2\rangle\,\langle 2\, 3\rangle\cdots\langle n\, 1\rangle}\, V_n\,, \tag{15}$$

where V_n is a universal, cyclically symmetric function,

$$V_{2m+1} = \sum_{r=1}^{m-1}\sum_{i=1}^{n} f_{i,r}\;, \qquad\qquad V_{2m} = \sum_{r=1}^{m-2}\sum_{i=1}^{n} f_{i,r} + \sum_{i=1}^{n/2} f_{i,m-1}\;. \tag{16}$$

The scalar box integral functions in $4-2\epsilon$ dimensions are given through $\mathcal{O}(\epsilon^0)$ by

$$\begin{aligned}
f_{i,r} = &-\frac{1}{\epsilon^2}\Big[(-t_{i-1}^{[r+1]})^{-\epsilon} + (-t_i^{[r+1]})^{-\epsilon} - (-t_i^{[r]})^{-\epsilon} - (-t_{i+r+1}^{[n-r-2]})^{-\epsilon}\Big] - \mathrm{Li}_2\Bigg(1 - \frac{t_i^{[r]}t_{i+r+1}^{[n-r-2]}}{t_{i-1}^{[r+1]}t_i^{[r+1]}}\Bigg) \\
&+ \mathrm{Li}_2\Bigg(1 - \frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\Bigg) + \mathrm{Li}_2\Bigg(1 - \frac{t_i^{[r]}}{t_i^{[r+1]}}\Bigg) + \mathrm{Li}_2\Bigg(1 - \frac{t_{i+r+1}^{[n-r-2]}}{t_{i-1}^{[r+1]}}\Bigg) + \mathrm{Li}_2\Bigg(1 - \frac{t_{i+r+1}^{[n-r-2]}}{t_i^{[r+1]}}\Bigg) + \frac{1}{2}\ln^2\Bigg(\frac{t_{i-1}^{[r+1]}}{t_i^{[r+1]}}\Bigg)
\end{aligned} \tag{17}$$

where $t_i^{[r]} = (k_i + k_{i+1} + \cdots + k_{i+r-1})^2$. Since $t_i^{[1]} = k_i^2 = 0$, we set $(-t_i^{[1]})^{-\epsilon} = 0$ in (17).

6. Conclusions

We have argued that the use of supersymmetry and string theory (the latter more heuristically), in combination with more conventional tools such as helicity and color decompositions, unitarity and collinear limits, can lead to many simplifications in the calculation of one-loop multi-parton amplitudes. At the practical level, some of these tools have been instrumental in calculating the one-loop five-parton amplitudes ($ggggg$, $\bar{q}q\bar{q}qg$ and $\bar{q}qggg$) which form the analytical bottleneck to NLO cross-sections for three-jet events at hadron colliders.[5,6,7] They have also been used to obtain infinite sequences of special one-loop helicity amplitudes in closed form.[25,28,17,24] The polynomial ambiguities in the non-supersymmetric components of one-loop QCD amplitudes are the main obstacle to their efficient evaluation. If one can show that these ambiguities may be fixed uniquely (and efficiently!) using factorization limits, then this obstacle would be lifted, and one would have a general technique for constructing one-loop QCD amplitudes without ever evaluating genuine loop diagrams.

7. References

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